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ON DETERMINING THE RELATION BETWEEN THE MEAN STRESS AND DEFORMATION TENSORS IN STRUCTURALLY INHOMOGENEOUS ELASTIC MEDIA*

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The paper deals with establishing a rheological equation connecting the mean stress and deformation tensors in a structurally inhomogeneous elastic medium.The problem is formulated in the dynamic terms, and the defining equations are sought in the form of linear operator relations. The expression for the elastic operator is obtained in the form of a perturbation series, the terms of which represent a superposition of the linear integral operators. Partial sums of the series are obtained using the method of changing the field variables (taking into account the multiple scattering). An expression for the elastic operator kernel is obtained consisting of three different type terms, namely the quasi-static, couple stress and nonlocal terms.

The problem of determining the relation between the mean stress and deformation tensors was formulated in /1/, where the connection with the couple stress theories was established under certain conditions. The existence of such a connection was first shown in /2/. In general, the relation between the macroscopic stresses and deformations is an operator-type relation /3—5/, and in the case of static problems it is sufficient to compute only the quasi-static part in the one-point approximation /6—8/. In the dynamic problems the two-point and multi-point approximations are important, because they determine the decay and dispersion of the waves investigated in the Born approximation in /7—10/. The elastic operator is computed, with the scattering of every multiplicity taken into account, using the method of changing the field variables /5,11/. The kernel of the operator is written in explicit form and this allows one to consider, depending on the characterisitic scale of the field variations, the connection with the couple stress and nonlocal theories.

1. Let us consider an elastic medium in which the stresses and deformations are connected by the generalized Hooke's law

$$\sigma = \lambda e, \quad e_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i}) \tag{1.1}$$

Here and in the following the straightforward tensor notation is used, reverting to the indicial notation, whenever necessary to find their value, and $\lambda_{ijkl}(\mathbf{x})$ is the elastic coefficients tensor depending randomly on the spatial coordinates.

The propagation of a harmonic wave is a randomly inhomogeneous elastic medium is described by the equation

$$Lu = 0, \quad L = \partial\Lambda\partial + \rho_0\omega^2 \tag{1.2}$$

The relations (1.1) and (1.2) are local and statistically nonlinear, therefore the system of equations in moments will not be closed. Let us introduce an elastic operator Λ^* by means of the relation

$$\langle\sigma\rangle = \langle\lambda e\rangle = \Lambda^* \langle e\rangle = \int \lambda^*(\mathbf{x}, \mathbf{x}_1)\langle e(\mathbf{x}_1)\rangle\, d\mathbf{x}_1 \tag{1.3}$$

where Λ^* is a linear integral operator the kernel of which remains to be determined. Averaging (1.2) and taking (1.3) into account, we obtain

$$L^*\langle u\rangle = 0, \quad L^* = \partial\Lambda^*\partial + \rho_0\,\omega^2 \tag{1.4}$$

For higher order moments of the displacement field u, closed systems of equations can be obtained in the same manner by introducing effective higher order operators.

Let us turn our attention to the problem of computing the kernel $\lambda^*_\circ(\mathbf{x}, \mathbf{x}_1)$. To do this we introduce into our discussion an auxilliary medium with parameters $\overset{\circ}{\lambda}_{ijkl}, \rho_0$ the dynamics of which are described by the equation

$$L_0 u_0 = 0 \tag{1.5}$$

Equations (1.2), (1.5) are reduced to the equivalent integral equation

$$e = e_0 - \int G_{,\mathbf{xx}}(\mathbf{x} - \mathbf{x}_1)\,\lambda'(\mathbf{x}_1)\,e(\mathbf{x}_1)\,d\mathbf{x}_1, \quad G_{,\mathbf{xx}} = G^{(s)}\delta(\mathbf{x} - \mathbf{x}_1) + G^{(r)}(\mathbf{x} - \mathbf{x}_1) \tag{1.6}$$

*Prikl.Matem.Mekhan.,44,No.3,550-556,1980 where $G_{,xx}$ is the second order derivative of the dynamic Green's tensor of an unbounded medium with parameters λ_0, ρ_0, $G^{(s)}$ being its singular and $G^{(r)}$ its regular part.

The solution of (1.6) is written in the form of a series in powers of the tensor $\lambda'_{ijkl}(\mathbf{x})$, and the n-th term of the series corresponds to n-tuple scattering of the field u_0 on the inhomogeneities of the medium. The expression for $\lambda^*(\mathbf{x}, \mathbf{x}_1)$ has the form of a series in moments of the field $\lambda(x)$. So far, only the two-point moments are known in the explicit form, therefore we usually have to limit ourselves to the correlation (Born) approximation with corresponds to a single scattering. This approximation presupposes the smallness of the fluctuations λ', i.e. it is useful for weakly inhomogeneous media.

Let us consider the problem of computing $\lambda^*(\mathbf{x}, \mathbf{x}_1)$ using the method of replacing the field variables /5/. We introduce, in place of $\lambda(\mathbf{x})$ and $e(\mathbf{x})$, new field variables $\gamma(\mathbf{x})$ and $E(\mathbf{x})$ according to the formulas

$$\gamma_{njst} = \lambda'_{njkl} B^{-1}_{klst}, \quad E_{im} = B_{imkl} e_{kl}, \quad B_{imkl} = \delta_{ik}\delta_{ml} + G^{(s)}_{in,\,mj}\lambda'_{njkl} \tag{1.7}$$

$$G_1^{(s)} = (3\lambda_0 + 8\mu_0)\, m_0^{-1}, \quad G_2^{(s)} = -(\lambda_0 + \mu_0)\, m_0^{-1}, \quad m_0 = 15\mu_0(\lambda_0 + 2\mu_0)$$

Then the equations (1.6) become

$$E = e_0 - \int G^{(r)}(\mathbf{x} - \mathbf{x}_1)\, \gamma(\mathbf{x}_1)\, E(\mathbf{x}_1)\, d\mathbf{x}_1 \tag{1.8}$$

Solving (1.8) by consecutive iterations, we obtain a series in powers of $\boldsymbol{\gamma}(\mathbf{x})$ which converges most rapidly when

$$\langle \gamma_{ijkl} \rangle = 0 \tag{1.9}$$

We can verify directly that $\langle E \rangle$ satisfies the equation

$$\langle E \rangle = e_0 + \int\int G^{(r)}(\mathbf{x} - \mathbf{x}_1)\, Q(\mathbf{x}_2, \mathbf{x}_1)\, \langle E(\mathbf{x}_1)\rangle\, d\mathbf{x}_1\, d\mathbf{x}_2, \quad Q(\mathbf{x}_1, \mathbf{x}_2) = \langle \gamma(\mathbf{x}_1)\, \gamma(\mathbf{x}_2)\rangle\, G^{(r)}(\mathbf{x}_1 - \mathbf{x}_2) + \ldots \tag{1.10}$$

Averaging (1.8) we obtain

$$\langle E \rangle = e_0 - \int G^{(r)}(\mathbf{x} - \mathbf{x}_1)\, \Gamma^*(\mathbf{x}, \mathbf{x}_1)\, d\mathbf{x}_1, \quad \langle \gamma E \rangle = \Gamma^* \langle E \rangle = \int \gamma^*(\mathbf{x}, \mathbf{x}_1)\, \langle E(\mathbf{x}_1)\rangle\, d\mathbf{x}_1 \tag{1.11}$$

and equating (1.10) with (1.11) we find

$$\gamma^* = -Q \tag{1.12}$$

In what follows, we shall assume the medium to be statistically isotropic and homogeneous.

Let us average (1.7), taking into account the expressions defining the operators Λ^* and Γ^*, and apply the Fourier transformation. Then the relation between the transforms

$$\Pi^*(\omega, \mathbf{k}) = \int \lambda^*(\rho)\, e^{-i\mathbf{k}\rho}\, d\rho, \quad D^*(\omega, \mathbf{k}) = \int \gamma^*(\boldsymbol{\rho})\, e^{-i\mathbf{k}\rho}\, d\boldsymbol{\rho}$$

is defined by the formulas

$$\Pi^*_{stkl} = \overset{\circ}{\lambda}_{stkl} + M^{-1}_{stnj} D^*_{njkl}, \quad M_{pnqj} = \delta_{pn}\delta_{qj} - D^*_{njim} G^{(s)}_{ip,\,mq} \tag{1.13}$$

Applying the inverse Fourier transformation to the relations (1.13), we obtain the expression for the kernel of the elastic operator in the form

$$\lambda^*(\omega, \boldsymbol{\rho}) = \lambda^\circ\, \delta(\boldsymbol{\rho}) + \lambda^{(d)}(\omega, \boldsymbol{\rho}), \quad \boldsymbol{\rho} = \mathbf{x} - \mathbf{x}_1, \quad \lambda^{(d)}(\omega, \boldsymbol{\rho}) = (2\pi)^{-3} \int M^{-1}(\omega, \mathbf{k})\, D^*(\omega, \mathbf{k})\, e^{i\mathbf{k}\rho} d\mathbf{k} \tag{1.14}$$

From (1.14) it follows that the kernel of the elastic operator is written in the form of a sum of the singular part with the tensor $\overset{\circ}{\lambda}_{ijkl}$ of elastic constants obtained from (1.9), and the dynamic part $\lambda^{(d)}_{ijkl}(\omega, \boldsymbol{\rho})$ which, in general, represents a series in moments $\gamma_{ijkl}(\mathbf{x})$.

2. Let the medium in question be a two-component composite with isotropic components. Then the relations (1.9) are reduced to the form

$$8\, G_0^3 - G_0^2\, [20\, \langle G \rangle - 9K_0 - 12\, \langle G_1 + G_2 \rangle] - 3G_0\, [5K_0\, \langle G \rangle - 2K_0 S_g + 4G_1 G_2] - 6K_0 G_1 G_2 = 0 \tag{2.1}$$

$$K_0 = \langle K \rangle + \frac{R_k(0)}{{}^4/_3 G_0 + S_k}, \quad S_f = c_2 f_1 + c_1 f_2, \quad c_1 + c_2 = 1, \quad R_f(0) = c_1 c_2 (f_1 - f_2)^2$$

where G_0 and K_0 are the macroscopic shear and volume moduli of elasticity of the auxilliary equivalent medium /5/.

Let us now turn to computing $\lambda^{(d)}(\omega, \boldsymbol{\rho})$. We note that the tensors $\Pi^*_{nj\gamma\delta}(\omega, \mathbf{k})$, $D^*_{nj\gamma\delta}(\omega, \mathbf{k})$ in the space of wave numbers $\mathbf{k} = \{k_i\}$ $(i = 1, 2, 3)$ and $\lambda^{(d)}_{nj\gamma\delta}(\omega, \boldsymbol{\rho})$, $\gamma^*_{nj\gamma\delta}(\omega, \rho)$ in the Cartesian coordinate space $\boldsymbol{\rho} = \{\rho_i\}$ $(i = 1, 2, 3)$ have the form of isotropic tensors /5,11/

$$N_{nj\gamma\delta}(\omega, \mathbf{y}) = \sum_{\alpha=1}^{6} N_\alpha\ (\omega, y)\, I^\alpha_{nj\gamma\delta}\ (\mathbf{y} = \mathbf{k}, \boldsymbol{\rho})$$

$$I^1_{nj\gamma\delta} = \delta_{nj}\delta_{\gamma\delta}, \quad I^2_{nj\gamma\delta} = {}^1/_2\, (\delta_{n\gamma}\delta_{j\delta} + \delta_{n\delta}\delta_{j\gamma}), \quad I^3_{nj\gamma\delta} = y_n^\circ y^\circ_j \delta_{\gamma\delta}, \quad I^4_{nj\gamma\delta} = y_\gamma^\circ y_\delta^\circ \delta_{nj}$$

$$I^5_{nj\gamma\delta} = {}^1/_4\,(\delta_{n\gamma}y_j^\circ y_\delta^\circ + \delta_{n\delta}y_j^\circ y_\gamma^\circ + \delta_{j\gamma}y_n^\circ y_\delta^\circ + \delta_{j\delta}y_n^\circ y_\gamma^\circ),\quad I^6_{nj\gamma\delta} = y_n^\circ y_j^\circ y_\gamma^\circ y_\delta^\circ,\quad y_i^\circ = y_i\,|y|^{-1}$$

We shall compute $\lambda^{(d)}(\omega, \rho)$ in the two-point approximation in $\gamma(\mathbf{x})$ which corresponds to summing, in the dispersion series over $\lambda(\mathbf{x})$, of the two-point moments of all orders, i.e. to taking into account the scatterings of every multiplicity.

For the strongly isotropic media $N_\alpha = 0$ $(\alpha = 3, 4, 5, 6)$ /5/, and the formulas (1.13) become

$$\Pi_2^* = G_0 + \frac{D_2^*}{1 - T^{(t)}D_2^*},\quad T^{(t)} = \frac{2\,(3\lambda_0 + 8G_0)}{15G_0\,(\lambda_0 + 2G_0)} \tag{2.2}$$

$$K^* = K_0 + \frac{D^*}{1 - TD^*},\quad T = (\lambda_0 + 2G_0)^{-1},\quad K^* = \Pi_1^* + {}^2/_3\Pi_2^*,\quad D^* = D_1^* + {}^2/_3 D_2^*$$

The eigenvalues D_2^* and D^* can be written, for the exponential type correlation functions, in the form of series in powers of k^2

$$D^*(\omega, k) = \sum_{n=0}^{\infty} \gamma_n k^{2n},\quad D_2^*(\omega, k) = \sum_{n=0}^{\infty} \gamma_n^{(2)} k^{2n} \tag{2.3}$$

The coefficients γ_n and $\gamma_n^{(2)}$ are fairly bulky and depend on ω, K_0, G_0 and the correlation radius a. The character of the dependence is determined by the form of the actual correlation function.

Taking into account the expressions

$$\Phi(\omega, k) = 1 - T\sum_{n=0}^{\infty} \gamma_n k^{2n} = (1 - T\gamma_0)\prod_{n=0}^{\infty}(1 - k^2 k_n^{-2}),\quad \Phi^{(t)}(\omega, k) = 1 - T^{(t)}\sum_{n=0}^{\infty}\gamma_n^{(2)} k^{2n} = (1 - T^{(t)}\gamma_0^{(2)})\prod_{n=0}^{\infty}(1 - k^2 k_n^{(t)-2})$$

we write the formulas (2.2) in the form

$$\Pi_2^*(\omega, k) = G_0 + 2\sum_{n=0}^{\infty}\gamma_n^{(2)}k^{2n}\sum_{m=0}^{\infty}\frac{k_m^{(t)}}{\Phi^{(t)\prime}(k_m^{(t)})}\frac{1}{k^2 - k_m^{(t)2}},\quad K^*(\omega, k) = K_0 + 2\sum_{n=0}^{\infty}\gamma_n k^{2n}\sum_{m=0}^{\infty}\frac{k_m}{\Phi'(k_m)}\frac{1}{k^2 - k_m^2} \tag{2.4}$$

Here k_m and $k_m^{(t)}$ are roots of the functions $\Phi(\omega, k)$ and $\Phi^{(t)}(\omega, k)$, respectively.

Passing in (2.4) back to the originals in accordance with the formulas (1.14), we obtain the following expressions for the kernels of the elastic operator (Δ is the Laplace operator):

$$K^*(\omega, \rho) = K_0\delta(\rho) + \frac{1}{2\pi}\sum_{n=0}^{\infty}\gamma_n\Delta^n\Big(\frac{1}{\rho}\sum_{m=0}^{\infty}\frac{k_m}{\Phi'(k_m)}e^{ik_m\rho}\Big) \tag{2.5}$$

$$\lambda_2^*(\omega, \rho) = G_0\delta(\rho) + \frac{1}{2\pi}\sum_{n=0}^{\infty}\gamma_n^{(2)}\Delta^n\Big(\frac{1}{\rho}\sum_{m=0}^{\infty}\frac{k_m^{(t)}}{\Phi^{(t)\prime}(k_m^{(t)})}e^{ik_m^{(t)}\rho}\Big)$$

Using the representation

$$\Delta^n\Big(\frac{e^{ik\rho}}{\rho}\Big) = (-1)^n k^{2n}\frac{e^{ik\rho}}{\rho} + 4\pi\sum_{l=1}^{n}k^{2(n-l)}(-1)^{(n+1-l)}\Delta^{l-1}\delta(\rho)$$

we transform (2.5) to the form

$$K^*(\omega, \rho) = K_0\delta(\rho) + \sum_{n,\,m=0}^{\infty}\sum_{l=1}^{n}A_{nml}(\omega)\,\Delta^{l-1}\delta(\rho) + \sum_{n,\,m=0}^{\infty}\varkappa_{nm}(\omega)\frac{e^{ik_m\rho}}{\rho} \tag{2.6}$$

$$\Pi_2^*(\omega, \rho) = G_0\delta(\rho) + \sum_{n,\,m=0}^{\infty}\sum_{l=1}^{n}A_{nml}^{(t)}(\omega)\,\Delta^{l-1}\delta(\rho) + \sum_{n,\,m=0}^{\infty}\varkappa_{nm}^{(t)}(\omega)\frac{e^{ik_m\rho}}{\rho}$$

$$A_{nml}^{(\alpha)} = 2\gamma_n^{(\alpha)}\frac{k_m^{(\alpha)[2(n-l)+1]}}{\Phi^{(\alpha)\prime}(k_m^{(\alpha)})}(-1)^{(n+1-l)},\quad \varkappa_{nm}^{(\alpha)} = (-1)^n\frac{k_m^{(\alpha)(2n+1)}\gamma_n^{(\alpha)}}{2\pi\Phi^{(\alpha)\prime}(k_m^{(\alpha)})}$$

Substituting (2.6) into (1.3) we obtain, in the straightforward notation, the following expression connecting the mean stress and deformation tensors:

$$\langle\sigma\rangle = \lambda_0\langle e\rangle + \sum_{n,\,m=0}^{\infty}\sum_{l=1}^{n}A_{nml}(\omega)\,\Delta^{l-1}\langle e\rangle + \sum_{n,\,m=0}^{\infty}\varkappa_{nm}(\omega)\int\frac{\exp\,[ik_m|\mathbf{x} - \mathbf{x}_1|]}{|\mathbf{x} - \mathbf{x}_1|}\langle e(\mathbf{x}_1)\rangle\,d\mathbf{x}_1 \tag{2.7}$$

From (2.6) and (2.7) it follows that in the general case the elastic operator and the macroscopic stresses have the form of a sum containing a quasi-static part, a term depending on the even derivatives of mean deformations, and an operator part. Depending on the quantity ω and the scale of variation in $\langle e\rangle$, different terms become dominant in the defining relations (2.7). For the static fields $\omega = 0$ the quasi-static part will be essential, and the long waves (low frequencies) will demand the terms of the second part to be taken into account. The third term becomes essential for the wavelengths of the order of the inhomogeneities. For the rapidly oscillating fields $(\omega \to \infty)$ the dispersion vanishes.

3. Let us compute the kernel of the effective operator, using the standard equation $\Delta\varphi + k_0^2\mu_0\mu(x)\varphi = 0$ /12/ for the medium characterised by the correlation function

$$\langle\gamma(x_1)\gamma(x_2)\rangle = k_0^4\mu_0^2R(0)e^{-\rho a^{-1}}, \quad \gamma(x) = k_0^2\mu_0\mu'\mu^{-1}$$

The equation for the averaged field $\langle\varphi\rangle$ has the form

$$\Delta\langle\varphi\rangle + k_0^2\mu_0M^*\langle\varphi\rangle = 0 \tag{3.1}$$

and the integral operator M^* in (3.1) is defined by the relation

$$\langle\mu\varphi\rangle = M^*\langle\varphi\rangle = \int\mu^*(\mathbf{x}-\mathbf{x}_1)\langle\varphi(\mathbf{x}_1)\rangle\, d\mathbf{x}_1$$

The eigenvalue $m^*(\omega, k)$ of the operator M^* is given, analogously to (1.13) and (2.2), in terms of the eigenvalue $g^*(\omega, k)$ of the operator Γ^*, by the formula

$$m^*(\omega, k) = \frac{\mu_0}{1 - T_0g^*}, \quad T_0 = (\mu_0k^2)^{-1}, \quad g^*(\omega, k) = \frac{\pi k_0^4\mu_0^2R(0)}{2(\alpha^2+k^2)}, \quad \alpha = (1 - iak_0\mu_0^{1/2})\, a^{-1} \tag{3.2}$$

Passing in (3.2) to the originals, we obtain an expression for the kernel $\mu^*(x - x_1)$ in the form

$$\mu^*(x - x_1) = \mu_0\delta(x - x_1) + \frac{\pi R(0)k_0^2\mu_0}{2}\frac{e^{ik_*\rho}}{\rho}, \quad k_*^2 = \frac{\pi R(0)k_0^2\mu_0}{2} - \alpha^2 \tag{3.3}$$

Let us write k_* as follows:

$$k_* = z(1 + \pi k_0^2\mu_0R(0)2^{-1}z)^{1/2} \approx z\left(1 - \pi R(0)a^2k_0^2\mu_0 4^{-1}\sum_{k=0}^{\infty}(k+1)(iak_0\mu^{1/2})^k + \ldots\right), \quad z = k_0\mu_0^{1/2} + ia^{-1} \tag{3.4}$$

Then, putting $k_* \approx k_0\mu_0^{1/2} + ia^{-1}$, we obtain the Born approximation for $\mu^*(x - x_1)$ in (3.3). Inclusion of multiple scattering in the two-point approximation requires that all terms of the series (3.4) be taken into account, and the series are in this case summed.

The function $g^*(\omega, k)$ can be expanded within its circle of convergence into the series

$$g^*(\omega, k) = \frac{\pi k_0^4\mu_0^2R(0)}{2\alpha^2}\sum_{n=0}^{\infty}\left(-\frac{1}{\alpha^2}\right)^n k^{2n} \tag{3.5}$$

In accordance with (3.5), we write the kernel of the operator M^* in the form

$$\mu^*(x - x_1) = \frac{\mu_0}{(1 - T\gamma_0)4\pi\rho}\sum_{n=0}^{\infty}\frac{k_m(\omega)}{\Phi'(k_m(\omega))}e^{ik_m\rho} \tag{3.6}$$

From (3.6) it follows that the kernel μ^* contains a local term and an operator term, but not a couple stress term.

We note that the acoustic approximation discussed, which is defined by $g^*(\omega, k)$, is contained within the expression (2.2) for D^* and characterizes the longitudinal wave. The term determines the domain of convergence of D^*. Moreover, for the high frequencies the equations of elasticity of the inhomogeneous medium reduce to two Helmholtz equations /13/ and this enables us, using the results of the investigation of the standard equation, to judge certain effects occurring in the case of general equations of a structurally inhomogeneous elastic medium.

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